Approximating Solutions of Nonlinear Abstract Measure First Order Differential Equations via Hybrid Fixed Point Theory

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Abstract: *In this article we prove the existence and approximations of solutions of AMDE of first order ordinary nonlinear hybrid differential equations.*

We relay our results on Dhage iterations principle or method embodied in a recent hybrid fixed point theorem of Dhage [B.C. Dhage, on some nonlinear alternatives of Leray-Schauder type and functional integral equations, Arch. Math. (Brno) 42 (2006) 11-23] under the mixed generalized Lipschitz and Caratheodory Conditions.

The existence and approximations of solutions is also proved under certain monotonicity conditions and using a hybrid fixed point theorem of Dhage given in the above mentioned reference on ordered Banach spaces.

Keywords: *Abstract measure differential equation, hybrid fixed point theory, approximate Solution.*

1. Preliminaries:

Let X denotes a Banach space with an order relation $\leq$ and the norm $\|\,.\,\|$. It is known that X is regular if $\{x_n\}$ is a non decreasing (respectively non increasing) sequence in X such that $x_n \to x^*$ as $n \to \infty$, then $x_n \leq x^*$ (respectively $x_n \geq x^*$) for all $n \in N$. The conditions guaranteeing the regularity of X may be found in Heikkila and Lakshikantham [16] and the references there in we need the following definitions in the sequel.

Definition 1.1: A mapping $T : X \to X$ is called isotone or monotone non decreasing if it preserves the order relation $\leq$, that is, if $x \leq y$ implies $T_x \leq T_y$ for all $x, y \in X$. similarly, T is called monotone nonincreasing if $x \leq y$ imlies $T_x \geq T_y$ for all $x, y \in X$. Finaly, T is called monotone or simply monotone if it is either monotone nondecreasing or monotone nonincreasing on X.

An operator T on a Banach space X into itself is called compact if $T(X)$ is a relatively compact subset of X, T is called totally bounded if for any bounded subset S of X, $T(S)$ is a relatively compact subset of X. If T is continuous and totally bounded, then it is called completely continuous on X.

Definition 1.2: A mapping $T : X \to X$ is called partially continuous at a point $a \in X$ if for $\varepsilon > 0$ there exist a $\delta > 0$ such that $\|T_x - T_a\| < \varepsilon$, whenever x is comparable to a and $\|x - a\| < \delta$, T is called partially continuous on X if it partially continuous at every point of it. It is clear that if T is partially continuous on X then it is continuous on every chain C contained in X.

Definition 1.3: An operator T on a partially Banach space X into itself is called partially bounded if $T(C)$ is bounded for every chain C in X. T is called uniformly partially bounded if all chains $T(C)$ in X are bounded by a unique constant. T is called partially compact if $T(C)$ is a relatively compact subset of X for all totally ordered sets or chains C in X. T is called partially totally bounded if for any totally ordered and bounded subset C of X, $T(C)$ is a relatively compact subset of X. If T is partially continuous and partially totally bounded, then it is called partially completely continuous on X.

Remark 1.1: Note that every compact mapping on a partially Banach space is partially compact and every partially compact mapping is partially totally bounded, however the reverse implications donot hold, again, every completely continuous mapping is partially completely continuous and every partially completely continuous and partially totally bounded, but the converse may not be true.

Definition 1.4: The order relation $\leq$ and the metric d on a non-empty set X are said to be compatible if $\{x_n\}$ is a monotone ; that is , monotone nondecreasing or monotone nonincreasing sequence in X and if a subsequence $\{x_{nk}\}$ of $\{x_n\}$ converges to x^* implies that the whole sequence $\{x_n\}$ converges to x^* , similarly, given a partially Banach $(X, \leq, \|.\|)$, the order relation $\leq$ and the norm $\|.\|$ are said to be compatible if $\leq$ and the metric d defined through the norm $\|.\|$ are compatible.

Clearly, the set IR of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function has this property, similarly, the finite dimensional Euclidean space IR^n with usual component wise order relation and the standard norm possesses the compatibility property.

Definition 1.5: A mapping $T: X \to X$ is called ID- Lipschitz if there exists a continuous and nondecreasing function $\phi: IR^+ \to IR^+$ such that

$$\|T_x - T_y\| \leq \phi\|x - y\| \tag{1.1}$$

For all $x, y \in X$, where $\phi(0) = 0$. In particular if $\phi(r) = \alpha r, \alpha > 0,$ T is called a Lipschitz constant α . Further if $\alpha < 1$, then T is called a contraction on X with the contraction constant α .

Let X be Banach space and let $T: X \to X, T$ is called compact if $\overline{T(X)}$ is a compact subset of X . T is called totally bounded if for any bounded subset X . $T(S)$ is a totally bounded subset of X . $T(S)$ is a totally bounded subset of X . T is called completely continuous if T is continuous and totally bounded on X. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of X , The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [15].

The Dhage iteration principle or method (in short DIP or DIM) developed in Dhage [11,12,13,14] may be formulated as *" monotone convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation"* and which is powerful tool. In the existence theory of nonlinear analysis. It is clear that Dhage iteration method is different from the usual Picard's successive iteration method and embodied in the following application hybrid fixed point theorems proved in Dhage [13] which forms a useful key tool for our work contained in this paper. A few other hybrid fixed point theorem involving the Dhage iteration method may be found in Dhage [11.12,13,14].

2. Statement of the problem.

Let X be a real Banach algebra with a convenient norm $\|.\|$. Let $x, y \in X$ then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{z \in x \mid z = x + r(y - r), 0 \leq r \leq 1\} \tag{2.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$ then for any $x \in \overline{x_0 z}$, we define the sets Sx and $\bar{S}x$ in X by

$$Sx = \{rx \mid -\infty < r < 1\} \tag{2.2}$$

and

$$\bar{S}x = \{rx \mid -\infty < r \leq 1\} \tag{2.3}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary . we say $x_1 < x_2$ if $Sx_1 \subset Sx_2$ or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 z_2}$, In this case we also write $x_2 > x_1$.

Let M denote the $\sigma-$algebra of all subsets of X such that (X, M) is a measurable space. Let $ca(X, M)$ be the space of all vector measures (real signed measures) and define norm $|.|$ on $ca(X, M)$ by

$$\|p\| = |p|(X) \tag{2.4}$$

Where $|p|$ is a total variation measure of p and is given by

$$|p|(X) = \sup \sum_{i=1}^{\infty} |p(E_i)|, \qquad E_i \subset X, \tag{2.5}$$

Where the supremum is taken over all possible partitions $\{E_i : i \in N\}$ of X . it is known that $ca(X, M)$ is a Banach space with respect to the norm $\|.\|$ given by (2.4)

Let μ be a $\sigma-$ finite positive measure on X, and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E)=0$ implies $p(E)=0$ for some $E \in M$. in this case we also write $p \ll \mu.$.

Let $x_0 \in X$ be fixed and Let M_0 denote the σ-algebra on $S_{x0.}$, Let $z \in X$ be such that $z > x_0$ and M_z denote the $\sigma-$ algebra of all containing M_0 and the sets of the form S_x, $x \in \overline{x_0 z}$.

Consider the abstract measure differential equation (AMDE) of second of the form

$$\left.\begin{aligned} &\frac{dp}{d\mu} = f\left(x, p\left(\bar{S}_x\right)\right) + g\left(x, g\left(\bar{S}_x\right)\right) \qquad a.\ e.\ [\mu]\ on\ \overline{x_0\, z} \\ &p(E) = q(E), \qquad E \in M_0 \end{aligned}\right\} \tag{2.6}$$

Where q is a given known vector measure, $\frac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ, $f, g : S_z \times IR \to IR$ and $f\ (x, p(S_x))$ and $g(x, p(S_x))$ is $\mu-$integrable for each $p \in ca\ \ (S_z, M_z)$.

3. MAIN RESULTS

Theorem 3.1 : Let $(X, \leq, \|.\|)$ be a Banach space such that the order relation $\leq$ and the norm $\|.\|$ are compatible in X. Let $A, B : X \to X$ be two nondecreasing operators such that

(a) A is partially bounded and partially nonlinear $ID-$contraction.

(b) B is partially continuous and partially compact, and

(c) there exists an element $x_0 \in X$ such that $x_0 \leq Ax_0 + Bx_0 \quad or \quad x_o \geq Ax_0 + Bx_o$.

Then the operator equation $Ax + Bx = x$ has a solution x^* in X and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = Ax_n + Bx_n,\ n = 0, 1, \ldots$..converges monotonically to x^*.

Remarks 3.1: The conclusion of the theorem 3.1 also remains true if we replace the compatibility of X with respect to the order relation $\leq$ and the norm $\|.\|$ by a weaker condition of the compatibility of every compact chain C in X with respect to the order relation $\leq$ and the norm $\|\ .\ \|$. The later condition holds in particular if every partially compact subset of X possesses the compatibility property.

The equivalent integral formulation AMDE (2.6) is considered in the function space $ca(S_z, M_z)$ of continuous real valued function defined in $\overline{x_0\, z}$. we define a norm $\|\ .\ \|$ and the order relation '$\leq$'
in $ca(S_z, M_z)$ by

$$\|x\| = \sup_{t \in \overline{x_0\, z}} \|x(t)\| \tag{3.1}$$

$$x \leq y \Leftrightarrow x(t) \leq y\ (t) \tag{3.2}$$

For all $t \in \overline{x_0\, z}$. Cleary $ca(S_z, M_z)$ in a Banach space with respect to above supremum norm and also partially ordered with respect to the above partially order relation $\leq$. it is known that the partially ordered Banach space $ca(S_z, M_z)$ has some nice properties with respect to the above order relation in it. The following lemma follows by an application of Arzela-Ascoli theorem.

Lemma 3.1: Let $(ca\ \ (S_z, M_z), \leq, \|.\|)$ be a partially ordered Banach Space with the norm $\|.\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively, then $\|.\|$ and $\leq$ are comparable in every partially compact subset of $ca(S_z, M_z)$.

Proof: Let S be a partially compact subset of $ca(S_z, M_z)$ and let $\{x_n\}$be a monotone nondecreasing sequence of a points in S, Then we have

$$x_1(t) \le x_2(t) \le ---- \le x_n(t) \le ---- \text{ for each } t \in M_{Z+}$$

Suppose that a subsequence $\{x_{nk}\}$ of $\{x_n\}$ is convergent and converges to a point x in S, then the subsequence $\{x_{nk}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent By monotone characterization the whole sequence $\{x_n(t)\}$ is convergent and converges to a point $x(t)$ in M_z for each $t \in M_{z+}$. This show that the sequence $\{x_n(t)\}$ converges to $x(t)$ pointwise in S. To show the convergence is uniform, it is enough to show that the sequence is equicontinous , since S is partially compact every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzela-Ascoli theorem, Hence $\{x_n\}$ is convergent and converges uniformly to x, as a result $\|.\|$ and $\le$ are compatible in S. This completes the proof.

Definition 3.1: A function $p \in ca(S_z, M_z)$ is said to be a lower solution of the AMDE (2.6) if it satisfies

$$\left.\begin{aligned} &\frac{dp}{d\mu} \le f\left(x, p\left(\bar{S}_x\right)\right) + g\left(x, p\left(\bar{S}_x\right)\right) \\ &p(E) = q(E) \qquad , \qquad E \in M_0 \end{aligned}\right\} \tag{3.3}$$

Where q is a given known vector measure, similarly, an upper solution $v \in ca(S_z, M_z)$ for AMDE(2.6) is defined on $\overline{x_0\ z}$.

Definition 3.2: A function $\beta : S_z \times IR \to IR$ is called Caratheodory if

(I) $x \to \beta(x, y)$ is μ - measurable for each $y \in IR$, and

(II) $y \to \beta(x, y)$ is continuous almost everywhere $[\mu]$ on $\overline{x_0\ z}$

Further a Caratheodory function $\beta(x, y)$is called L^1_μ - Caratheodory if

(III) for each real number $r > o$ there exists a function $h_r \in L^1_\mu(S_z, IR)$ such that

$$|\beta(x, y)| \le h_r(x) \qquad a.e.[\mu], \quad x \in \overline{x_0 z} \quad \text{for all } y \in IR \text{ with} |y| \le r.$$

We consider the following set of assumptions

(A_1)There exist a $\mu-$integrable function $\alpha : S_z \to IR^+$ and $\lambda > 0$ with $\lambda \ge \alpha$ such that

$$0 \le [f(t, x) + \lambda x] - [f(t, y) + \lambda y] \le \alpha(x - y) \text{ for all } t \in \overline{x_0 z} \text{ and } x, y \in IR, \; x \ge y.$$

(B_1) There exists a constant $K_2 > 0$ such that $|g(t, x)| \le K_2$ for all $t \in \overline{x_0\ z}$ and $x \in IR$.

(B_2) $g(t, x)$ is nondecreasing in x for all $t \in \overline{x_0 z}$

(B_3) The AMDE (2.6) has a lower solution $u \in ca(S_z, M_z)$

Consider the boundary value problem

$$\left.\begin{aligned} &\frac{dp}{d\mu} + \lambda\, p(\mu) = \tilde{f}\left(x, p\left(\bar{S}_x\right)\right) + g\left(x, p\left(\bar{S}_x\right)\right) \\ &p(E) = q(E) \end{aligned}\right\} \tag{3.4}$$

For all $x \in \overline{x_0\ z}$, where q is a given known vector measure, $\frac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ. $\tilde{f}, g : S_z \times IR \to IR$, $f\left(x, p\left(\bar{S}_x\right)\right)$ and $g\left(x, p\left(\bar{S}_x\right)\right)$ is $\mu-$integrable for each $p \in ca(S_z, M_z)$and

$$\tilde{f}\left(x, p\left(\bar{S}_x\right)\right) = f\left(x, p\left(\bar{S}_x\right)\right) + \lambda x \tag{3.5}$$

Remark 3.2: A function $u \in ca\ (S_z, M_z)$ is a solution of the AMDE (3.4) iff it is a solution of the AMDE defined on $\overline{x_0\, z}$.

Consider the following assumptions

(A_2) There exists a constant $k_1 > 0$ such that $\left| \tilde{f}\left(x, p\left(\bar{S}_x\right)\right)\right| \leq k_1$ for all $x \in \overline{x_0\, z}$ and $x \in IR$.

The following useful lemma may be found in Torres [17]

Lemma 3.2: For any $h \in L^1_\mu(S_{z,} IR^+)$ and $\sigma \in L^1_\mu(S_z, IR^+)$, x is a solution to the differential equation

$$\left.\begin{aligned} &\frac{dp}{d\mu} + h(\mu)p(\mu) = \sigma(\mu) \qquad a.\,e.\,[\,\mu\,] \quad on\ \overline{x_0\, z} \\ &P(E) = q(E) \end{aligned}\right\} \tag{3.6}$$

If and only if it is a solution of the integral equation

$$p(\mu) = \int_E G_h\left(x, p\left(\bar{S}_x\right)\right)\sigma\left(\bar{S}_x\right)d\mu\,, \tag{3.7}$$

Where, $G_h\left(x, p\left(\bar{S}_x\right)\right)$ is a Green's function associated with the homogeneous boundary value problem.

$$\left.\begin{aligned} &\frac{dp}{d\mu} + h(\mu)p(\mu) = 0 \quad a.e.\ [\mu]\ on\ \overline{x_o z} \\ &p(E) = q(E) \end{aligned}\right\} \tag{3.8}$$

Notice that the Green's function G_h is continuous an nonnegative on $\overline{x_0\, z}$ and therefore, the number

$$M_h := \max\left\{\left| G_h\left(x, p\left(\bar{S}_x\right)\right)\right| : x,\ p\left(\bar{S}_x\right) \in \overline{x_0 z}\right\}$$

exists for all $h \in L^1_\mu(S_z, IR^+)$.

As an application of lemma (3.2) we obtain the following result.

Lemma 3.3: Suppose that hypotheses (A_2) and (B_1) hold, then a function $u \in ca(S_z, M_z)$ is a solution of the AMDE (3.4) iff it is a solution of the nonlinear integral equation

$$p(E) = \int_E G(t,x)\ \tilde{f}\left(x, p\left(\bar{S}_x\right)\right)d\mu + \int_E G(t,x)\, g\left(x,\ p\left(\bar{S}_x\right)\right)d\mu \tag{3.9}$$

for $t \in \overline{x_0\, z}$, where $G(t, x)$ is a Green's function associated with the homogeneous boundary value problem

$$\left.\begin{aligned} &\frac{dp}{d\mu}+\lambda\, p(\mu)=0\ , \quad t\in\overline{x_0\, z} \\ &p(E)=q(E) \end{aligned}\right\} \tag{3.10}$$

Theorem 3.2: Assume that $(A_1)-(A_2)$ and $(B_1)-(B_3)$ hold, Furthermore, if $\lambda M\,\mathrm{T}<1$, for some $\mathrm{T}>0$ then the AMDE (2.6) has a solution x^* defined on $\overline{x_0\, z}$ and the sequence $\{x_n\}$ of successive approximation defined by

$$p_{n+1}(E)=\int_E G(t,x)\tilde{f}\left(x, p\left(\bar{S}_x\right)\right)d\mu+\int_E G(t,x)\, g\left(x, p\left(\bar{S}_x\right)\right)d\mu \tag{3.11}$$

For all $t\in\overline{x_0\, z}$, where $p_0=u$ converges monotonically to x^*.

Proof: By lemma , every compact chain in $ca(S_z,M_z)$ is compatible with respect to the norm $\|.\|$ and order relation $\le$. Define two operators A and B on $ca(S_z,M_z)$.

Consider the open ball $B_r(0)$ in $ca(S_z,M_z)$ centred at origin $'o'$ of radius $'r'$, where $'r'$ is a positive real number satisfying (3.11). Define two operators $A: ca\,(S_z,M_z)\to ca\,(S_z,M_z)$, $B:\bar{B}_r(0)\to ca(S_z,M_z)$ by

$$Ap(E)=\begin{cases}\int_E G(t,x)\,\tilde{f}\left(x\, ,\, p\left(\bar{S}_x\right)\right)d\mu, & if\ E\in M_z,\ E\subset\overline{x_0 z}\\ 0, & if\ E\in M_0\end{cases} \tag{3.12}$$

And

$$Bp(E)=\begin{cases}\int_E G(t\, ,\, x)\, g\left(x, p\left(\bar{S}_x\right)\right)d\mu, & if\ E\in M_z,\ E\subset\overline{x_0 z}\\ G(t,x)\, q(E), & if\ E\in M_0\end{cases} \tag{3.13}$$

Now, by lemma, AMDE (3.4) is equivalent to the operator equation

$$Ap(E)+Bp\,(E)=p(E), \qquad E\subset\overline{x_0 z} \tag{3.14}$$

We show that the operators A and B satisfy all the conditions of theorem (3.1). this is achived in the series of following steps.

Step-I: A and B are nondecreasing operators on $ca(S_z,M_z)$.

Let $p_1,p_2\in ca(S_z,M_z)$ be such that $p_1\ge p_2$ then by hypothesis (A_1), we obtain

$$\begin{aligned} Ap_1(E) &= \int_E G(t,x)\, \tilde{f}\left(x, p_1(\bar{S}_x)\right) d\mu \\ &\geq \int_E G(t,x)\, \tilde{f}\left(x, p_2(\bar{S}_x)\right) d\mu \\ &= Ap_2(t) \end{aligned}$$

For all $t \in \overline{x_0\, z}$. This shows that A is nondecreasing operator on $ca(S_z, M_z)$ into $ca(S_{z,} M_z)$,similarly using hypothesis (B_2). It is shown that B is also nondecreasing on $ca(S_z, M_z)$ into itself. Thus A and B are nondecreasing operator on $ca(S_z, M_z)$ into itself.

Step-II: A is a partially bounded and partially contraction operator on $ca(S_z, M_z)$. Let $p \in ca(S_z, M_z)$ be arbitrary then by (A_2)

$$\begin{aligned} |Ap(E)| &\leq \left| \int_E G(t,x) \left| \tilde{f}\left(x, p(\bar{S}_x)\right) \right| d\mu \right| \\ &\leq \int_E G(t,x)\, k_1\, d\mu \\ &\leq M\, k\, \mathrm{T} \end{aligned}$$

For all $t \in \overline{x_0\, z}$. Taking the supremum over t in above inquality, we obtain $\| A_x \| \leq k_1 M\, \mathrm{T}$, and so, A is bounded. T is further implies that A is partially bounded on $ca(S_z, M_z)$.

Next, Let $p_1, p_2 \in ca(S_z, M_z)$ be such that $p_1 \geq p_2$

$$\begin{aligned} |Ap_1(E) - Ap_2(E)| &= \left| \int_E G(t,x) \left[\tilde{f}\left(x, p_1(\bar{S}_x)\right) - \tilde{f}\left(x, p_2(\bar{S}_x)\right) \right] \right| \\ &\leq \int_E G(t,x)\, \alpha \left[p_1(E) - p_2(E) \right] d\mu \\ &\leq \int_E G(t,x)\, \lambda\, |p_1(E) - p_2(E)|\, d\mu \\ &\leq \int_E \lambda M \| p_1 - p_2 \|\, d\mu \\ &= \lambda M\, \mathrm{T} \| p_1 - p_2 \| \end{aligned}$$

For all $E \in M_z$, Hence by definition of the norm in $ca(S_z, M_z)$, one has

$$\| Ap_1 - Ap_2 \| \leq \alpha \| p_1 - p_2 \|$$

For all $p_1, p_2 \in ca(S_z, M_z)$ with $p_1 \geq p_2$ where $0 \leq \alpha = \lambda M\, \mathrm{T} < 1$. Hence A is a partially contraction on $ca(S_z, M_z)$ which further implies that A is a partially continuous on $ca(S_z, M_z)$.

Step-III: B is a partially continuous operator on $ca(S_z, M_z)$,

Let $\{p_n\}$ be a sequence in $\overline{B}_r(0)$ converging to a vector measure p for all $n \in N$ converging to a vector measure p for all $n \in N$.

Then, By Dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty} Bp_n(E) &= \lim_{n\to\infty}\int_E G(t,x)\, g\left(x, p_n\left(\overline{S}_x\right)\right) d\mu \\ &= \int_E G(t,x)\left[\lim_{n\to\infty} g\left(x, p_n\left(\overline{S}_x\right)\right)\right] d\mu \\ &= \int_E G(t,x) g\left(x, p\left(\overline{S}_x\right)\right) d\mu \\ &= B_p(E)\end{aligned}$$

For all $t \in \overline{x_0 z}$. This shows that Bp_n converges to Bp pointwise on $\tilde{B}_r(0)$.

Next, we show $\{Bp_n\}$ is an equicontinuous sequence of functions in $ca(S_z, M_z)$. Let

$t_1, t_2 \in \overline{x_0 z}$ be arbitrary with $t_1 < t_2$.

Then

$$\begin{aligned}|Bp_n(t_2) - Bp_n(t_1)| &= \left|\int_E G(t_1, x) g\left(x, p_n\left(\overline{S}_x\right)\right) d\mu - \int_E G(t_2, x) g\left(x, p_n\left(\overline{S}_x\right)\right) d\mu\right| \\ &\le \left|\int_E |G(t_1, x) - G(t_2, x)|\left|g\left(x, p_n\left(\overline{S}_x\right)\right)\right| d\mu\right| \\ &\le \int_E |G(t_1, x) - G(t_2, x)|\, k_2\, d\mu \\ &\to 0 \text{ as } t_2 - t_1 \to 0\end{aligned}$$

Uniformly for all $n \in N$. This show that the convergence $Bp_n \to Bp$ is uniformly and hence B is partially continuous on $ca(S_z, M_z)$.

Step-IV: B is a partially compact operator on $ca(S_z, M_z)$.

Let C be an arbitrary chain in $ca(S_z, M_z)$. We show that $B(C)$ is a uniformly bounded and equicontinuous set in $ca(S_z, M_z)$. First we show that $B(C)$ is uniformly bounded . Let $p \in C$ be arbitrary, then

$$
\begin{aligned}
|Bp(E)| &= \left| \int_E G(t,x)\, g\left(x, p\left(\bar{S}_x\right)\right) d\mu \right| \\
&\le \int_E G(t,x) \left| g\left(x, p\left(\bar{S}_x\right)\right) \right| d\mu \\
&\le \int_E MK_2 \mathrm{T} = r
\end{aligned}
$$

For all $t \in \overline{x_0\, z}$. Taking supremum over t, we obtain

$\|Bp\| \le r$ for all $t \in C$. Hence B is a uniformly bounded subset of $ca(S_z, M_z)$. Next, we will show that $\mathrm{B}(C)$ is an equicontinuous set in $ca\,(S_z, M_z)$.

Let $t_1, t_2 \in \overline{x_0\, z}$ with $t_1 < t_2$ then

$$
\begin{aligned}
|\,Bp(t_2) - Bp(t_1)| &= \left| \int_E \left[G(t_1, x) - G(t_2, x) \right] g\left(x, p\left(\bar{S}_x\right)\right) d\mu \right| \\
&\le \int_E \left| G(t_1, x) - G(t_2, x) \right| \left| g\left(x, p\left(\bar{S}_x\right)\right) \right| d\mu \\
&\le \int_E \left| G(t_1, x) - G(t_2, x) \right| k_2\, d\mu \\
&\to 0 \quad as \;\; t_1 \to t_2
\end{aligned}
$$

Uniformly for all $p \in C$. Hence $\mathrm{B}(C)$ is compact subset of $ca(S_z, M_z)$ and consequently B is a partially compact operator on $ca(S_z, M_z)$ into itself.

Step-V: u satisfies the operator inequality $u \le Au + Bu$.

By hypothesis (H_4), the boundary value problem (1.1) has a lower solution u, then we have

$$
\left.
\begin{array}{lll}
\dfrac{dp}{d\mu} \le f\left(x, p\left(\bar{S}_x\right)\right) + g\left(x, p\left(\bar{S}_x\right)\right), & a.\,e.\,[\mu] & on \quad \overline{x_0\, z} \\
\quad and & & \\
p(E) \le q(E), & E \in M_0 &
\end{array}
\right\} \qquad (3.15)
$$

Integrating twice which together with the definition of the operator T implies that $u(\mu) \le Tu(\mu)$ for all $[\mu]$ on $\overline{x_0 z}$.

Consequently, u is a lower solution to be the operator equation $Tp = p$

Thus A and B satisfy all the condition of theorem (3.1)

With $p_0 = u$ and we apply it to conclude that the operator equation $Ap + Bp = p$ has a solution. Consequently the integral equation and AMDE (2.6) has solution x^* defined on $ca\,(S_z, M_z)$. Furthermore, the sequence $\{x_n\}$ of successive approximations defined by (3.4) converges monotonically to x^*.

This complete the proof.

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